Connection to Linear Transformations

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be any function.

The **kernel of T** is $\ker T = \{x \in \mathbb{R}^n \mid T(x) = 0\}$ (vectors that map to 0)

The **image/range** of T is $\text{im}\, T = \{T(x) \mid x \in \mathbb{R}^n\}$ (outputs)

Ex: A function from calculus $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 1$

Then $\ker f = \{1, -1\}$ since $x^2 - 1 = 0 \iff x = \pm 1$

and $\text{im} f = [-1, \infty)$ since $x^2 - 1 \geq -1 \quad \forall x \in \mathbb{R}$.

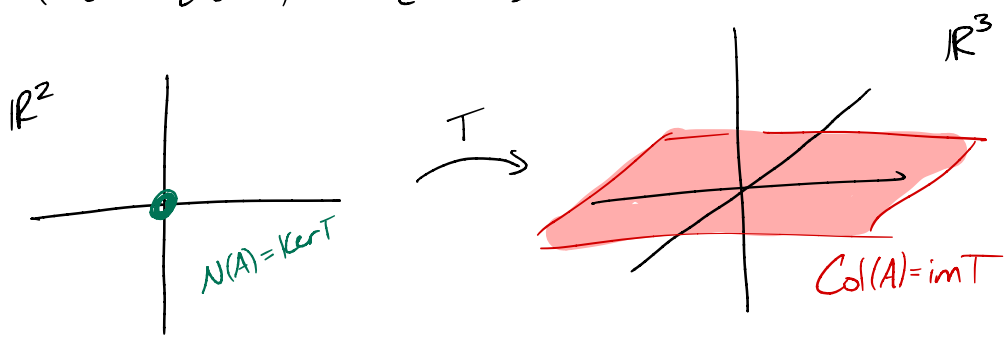
If T is a linear transformation with associated matrix A,

then $\ker T = N(A)$ and $\text{im}\, T = C(A)$.

$x \in \ker T \iff T(x) = 0 \iff Ax = 0 \iff x \in N(A)$

$y \in \text{im}\, T \iff y = T(x)$ for some x
$\iff y = Ax$ has soln
$\iff y \in C(A)$.

Ex: $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$



Solution Set of $Ax=b$

We can solve any system $Ax=b$ by reducing $[A|b]$ as we did for square matrices. If there is a pivot in the final column, then there is no solution. Otherwise, there are solutions. In this case, free variables $\Rightarrow$ ∞ sol'ns; no free variables $\Rightarrow$ 1 sol'n.

Ex: Let $A=\begin{bmatrix}1&1&2&4\\1&2&2&5\\1&3&2&6\end{bmatrix}$ and $b=\begin{bmatrix}3\\4\\5\end{bmatrix}$

To solve:

$$\left[\begin{array}{cccc|c}1&1&2&4&3\\1&2&2&5&4\\1&3&2&6&5\end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c}1&0&2&3&2\\0&1&0&1&1\\0&0&0&0&0\end{array}\right] \Rightarrow \begin{array}{l}x_1+2x_3+3x_4=2\\x_2+x_4=1\end{array}$$

(x_3, x_4 free)

We get the general solution:

$$\vec{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}-2x_3-3x_4+2\\-x_4+1\\x_3\\x_4\end{bmatrix} \quad \text{where } x_3, x_4 \in \mathbb{R}$$

We cannot express the solution set as a span.

$$\vec{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}2-2x_3-3x_4\\1-x_4\\x_3\\x_4\end{bmatrix}=\underbrace{\begin{bmatrix}2\\1\\0\\0\end{bmatrix}}_{\vec{x}_p}+x_3\begin{bmatrix}-2\\0\\1\\0\end{bmatrix}+x_4\underbrace{\begin{bmatrix}-3\\-1\\0\\1\end{bmatrix}}_{\vec{x}_n}$$

($\vec{x}_p$: particular soln when $x_3=x_4=0$)

There is a connection between this solution set and the solution set to $Ax=0$ (i.e. $N(A)$)

Last time, we found $N(A) = \text{Span}\left\{\begin{bmatrix}-2\\0\\1\\0\end{bmatrix}, \begin{bmatrix}-3\\-1\\0\\1\end{bmatrix}\right\}$

So a general solution is of the form:

$x = x_p + x_n$, where x_p is a particular sol'n and $x_n \in N(A)$.

This happens in general.

Let x_p be a solution to $Ax=b$ (if a sol'n exists)

Let x_n be a solution to $Ax=0$.

Then $A(x_p+x_n) = Ax_p + Ax_n = b+0 = b$

$\Rightarrow x_p+x_n$ is also a solution to $Ax=b$ (note $x_n=0$ is included)

Hence all vectors of the form x_p+x_n are solns to $Ax=b$.

On the other hand, let y be another solution of $Ax=b$

Then $A(y-x_p) = Ay - Ax_p = b-b = 0$

Hence $y-x_p$ is a solution to $Ax=0$ (or $y-x_p \in N(A)$)

So $y = x_p + (y-x_p)$ is the sum of x_p and a vector in $N(A)$

Hence $\{\text{Solutions of } Ax=b\} = \{x_p + x_n \mid x_p \text{ is a sol'n to } Ax=b \text{ and } x_n \in N(A)\}$

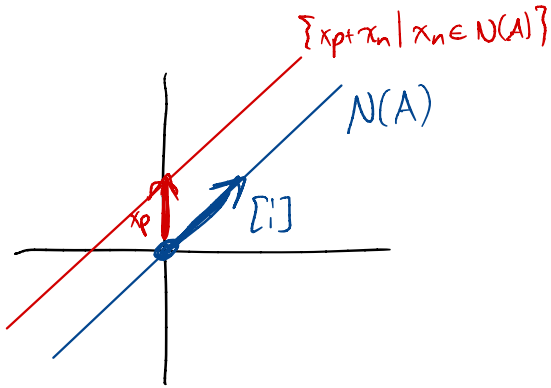
Geometric Picture

Recall, if A is $m \times n$, $N(A)$ is a subspace of $\mathbb{R}^n$. What does the solution set to $Ax=b$ look like (if it exists)?

Since the solution set is of the form $\{x_p + x_n \mid x_n \in N(A)\}$ it is $N(A)$ shifted in the direction of x_p

Ex: Suppose $N(A) = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ and $x_p = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Then:



Since $\{x_p + x_n \mid x_n \in N(A)\}$ doesn't pass through the origin (since $x_p \neq 0$), it is not a subspace

It is often called an affine subspace